Exploring the Set of APN Functions in Practice

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APN Functions

Almost Perfect Nonlinear Function

A function $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ is *Almost Perfect Non-linear* (APN) if $F(x + a) + F(x) = b$ has at most 2 solutions for all $a \neq 0, b$.

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- Example: $F(x) = x^3$ is APN for any n $\quad (x+a)^3 + x^3 = ax^2 + a^2x + a^3$

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- It is usually better for it to have a **higher degree**
- Sometimes, it needs to be **bijective**

The Big APN Problem

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Dillon Permutation (n = 6)

```
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[25 63 45 52  3 20 41 33]
[59 36  2 34 10  8 57 37]
[60 19 42 14 50 26 58 24]
[39 27 21 17 16 29  1 62]
[47 40 51 56  7 43 44 38]
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- The studied generalizations of the Kim mapping **do not** work
- Naive search not possible $2^6! = 2^{295}$

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$\hookrightarrow$ What do we mean by exploration :

1. The computation of the **CCZ class**
2. The computation of **Switching Neighbours**

Definitions - 1

$\mathbb{F}_{2^n} = \mathbb{F}_2^n$

We use heavily $\mathbb{F}_{2^n} = \mathbb{F}_2[X]/(P) = \mathbb{F}_2^n$ with $a = \sum_{i=0}^{n-1} a_i X^i = (a_0, \ldots, a_{n-1})$

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$F : \mathbb{F}_2^n \to \mathbb{F}_2^n$, $f : \mathbb{F}_2^n \to \mathbb{F}_2$

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Example

$F : \mathbb{F}_2^4 \to \mathbb{F}_2^4$, $x \mapsto x^3$

$$F = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} x_0x_2 + x_1x_2 + x_1x_3 + x_0 \\ x_0x_1 + x_0x_2 + x_2x_3 + x_3 \\ x_0x_1 + x_0x_2 + x_1x_2 + x_0x_3 + x_1x_3 + x_2x_3 + x_2 \\ x_1x_3 + x_2x_3 + x_1 + x_2 + x_3 \end{bmatrix}$$

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- **Multivariate Degree of** F : The maximum of the degrees of f_i
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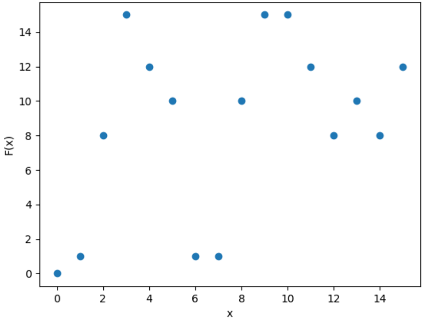
- **Univariate Degree of** F: The degree over $\mathbb{F}_{2^n}[X]$

$F(x) = x^3$ has multivariate degree 2 and univariate degree 3

Definitions - 3

$F : \mathbb{F}_2^n \to \mathbb{F}_2^n$

- **Graph of a Function**: $\Gamma_F = \{(x, F(x)) \mid x \in \mathbb{F}_2^n\}$



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The cube for $n = 6$

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For all EA-mappings

F is APN $\iff$ $B \circ F \circ A + C$ is APN, F and $B \circ F \circ A + C$ have same multivariate degree

Equivalence Relations - 2

Carlet-Charpin-Zinoviev (CCZ) Equivalence

F and G are *CCZ-equivalent* if there exist an affine bijection $\mathcal{A}$ over $\mathbb{F}_2^{2n}$ such that $\Gamma_G = \mathcal{A}(\Gamma_F)$, i.e :

$$\exists A, B, C, D \text{ affine}, \Gamma_G = \begin{bmatrix} A & D \\ C & B \end{bmatrix} (\Gamma_F)$$

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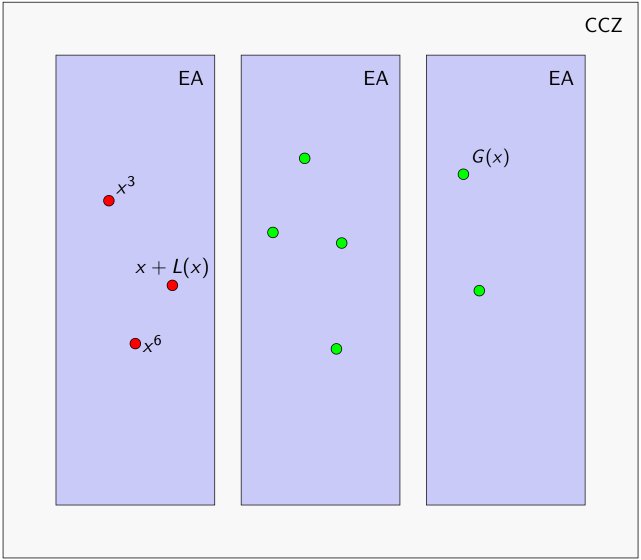
- A function is said to be **CCZ-quadratic** if it is CCZ-equivalent to a quadratic function

Equivalence - Summary

- Red Node: Quadratic function
- Green Node: Other degree
- Not all functions are represented

The cube for $n = 6$

- Exactly 3 EA-classes



Bestiary of APN CCZ-classes in numbers

A first lower bound

The number of CCZ classes of APN functions grows at least **exponentially** in the dimension (Kaspers and Zhou 2020).

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In 2025, in dimension 8

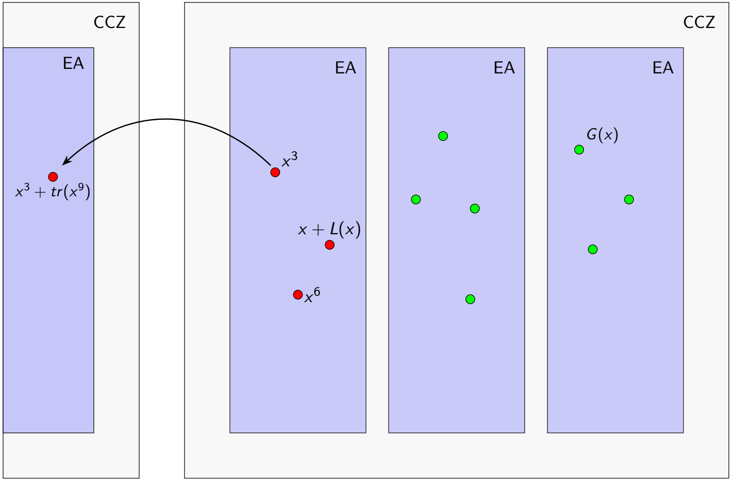
- Beierle et al. found 3.8 millions inequivalent quadratic functions...
- And they conjecture there are more !

What is a new function ?

- Red Node: Quadratic function
- Green Node: Other degree
- Not all functions are represented

A new function ?

With $tr(x) = \sum_{i=0}^{n-1} x^{2^i}$: $x^3 + tr(x^9)$ is not CCZ to x^3



Switching Neighbours

Switching Neighbours (Edel and Pott 2009)

$u \in \mathbb{F}_2^n$, g a Boolean function, $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ APN, the function $F + u \times g$ is called an APN Switching Neighbour of F if it is an APN function.

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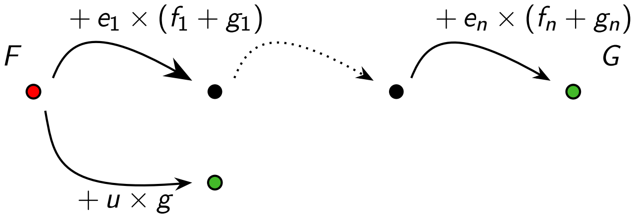
Since $F + G = \sum_{i=1}^{n} e_i \times (f_i + g_i)$:



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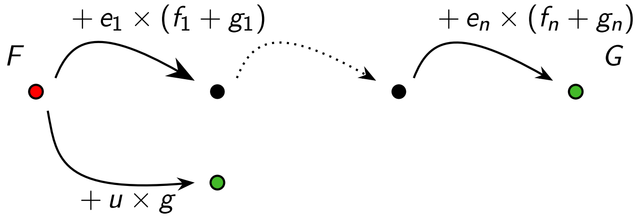
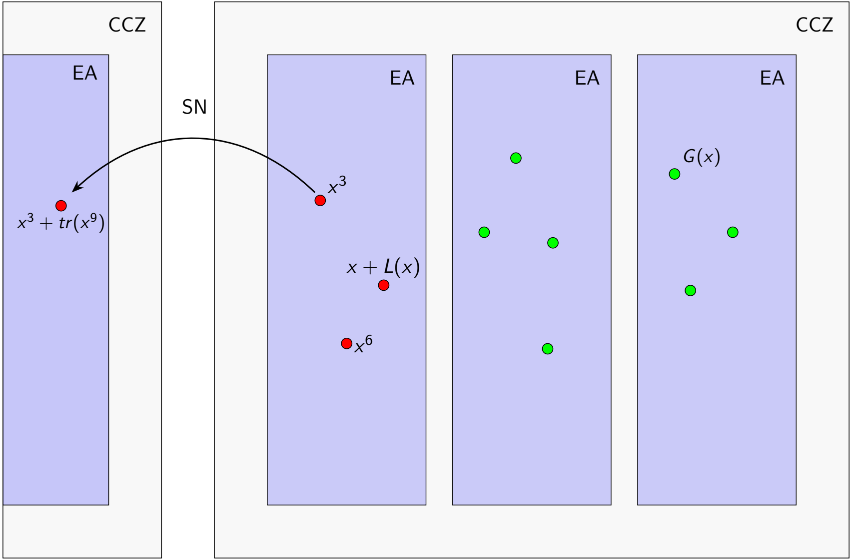


- Used to find the only known non-CCZ-quadratic APN in dimension 6 [EP09].
- Example: $x^3 + 1 \times tr(x^9)$, since $tr(x) = \sum_{i=0}^{n-1} x^{2^i}$ is a Boolean function

Exploration - Summary



Can we do it starting from the 3.8 millions function in dimension 8 ?

Could it be feasible ?

Problems

- Too many CCZ equivalent functions

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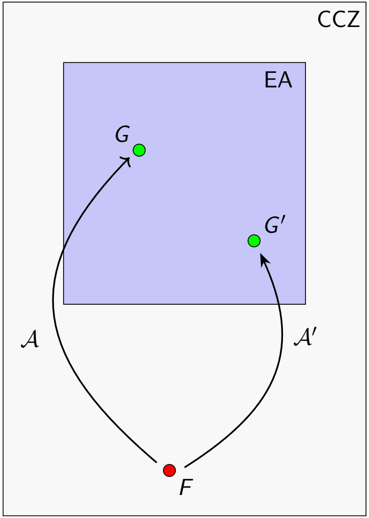
Our contributions

- We provide algorithms to efficiently explore CCZ-classes
- We provide theoretic improvements and algorithms to compute Switching Neighbours
- We provide public C++ implementations with SAGE bindings for all these algorithms

sboxU

- Library for S-box analysis, with all the implementations of this talk
- Available here: https://github.com/lpp-crypto/sboxU/

CCZ Class : What is the Problem ?



Fact

We know how to build admissible mappings, $\mathcal{A}, \mathcal{A}'$

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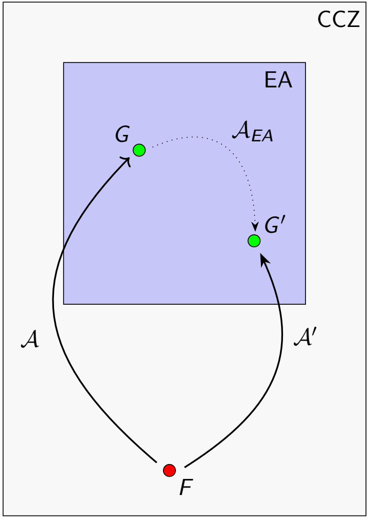

Fact

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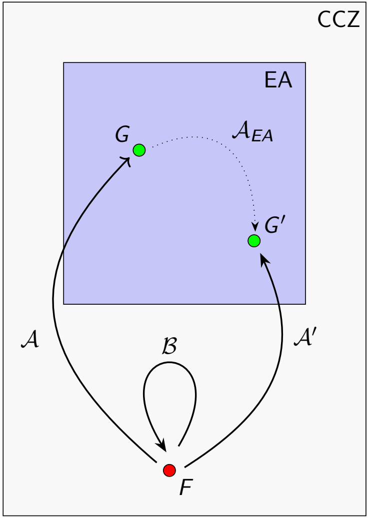

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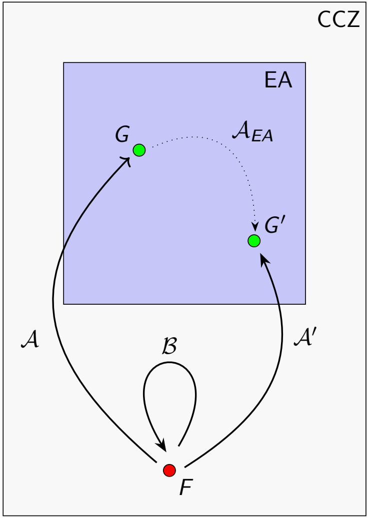

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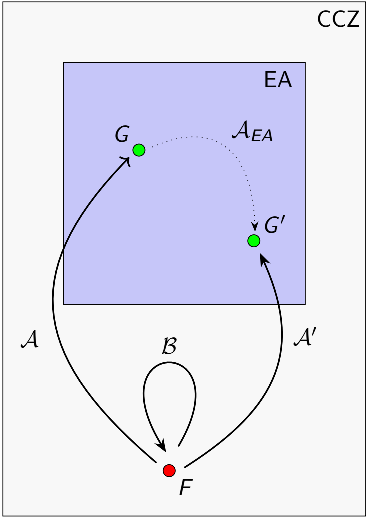

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We have to compute $\mathrm{Aut}(F)$ to quotient, but can we do it efficiently ?

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A lot of technical details

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CCZ class: Solutions

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- Tailored for quadratic functions, using an auxiliary function called ortho-derivative.
- The recovery involves spectral analysis with discrete Fourier transform

Takeaway

- We can compute efficiently $\mathrm{Aut}(F)$ for F quadratic
- We can compute one EA representative per EA class

Switching Neighbours: What is the Problem ?

Way to many uninteresting functions !

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Algorithmic problems

- Off-the-shelf algorithms are not efficient
- No filters for the uninteresting functions

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A lot of technical details

- We characterize the affine space of uninteresting functions

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Takeaway

- We gain a factor 2^{3n} on the number of switching neighbours that we need to check **per function**
- The new algorithm is 2^n times faster than the state of the art

CCZ classes for $\mathbb{F}_2^6$ and $\mathbb{F}_2^7$

$\mathbb{F}_2^6$

- We found all the 716 EA-classes in the union of the 14 known CCZ-equivalence classes (same as Calderini in 2020)

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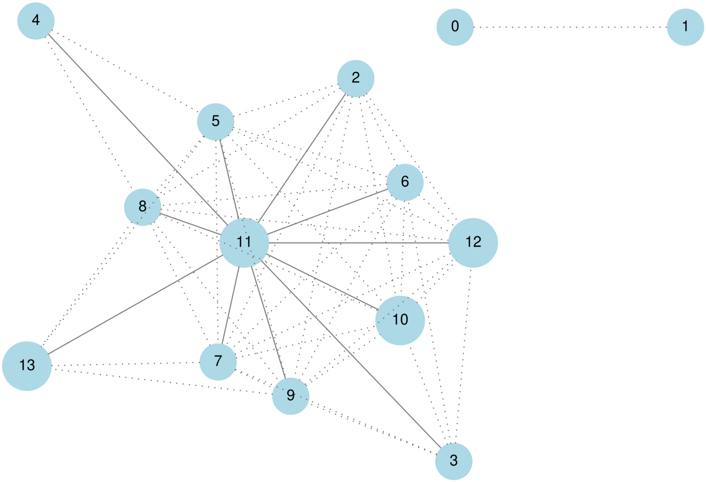
sboxU - For reproducibility

These classes are stored un tinySQL databases in sboxU !

For $\mathbb{F}_2^6$ - CCZ-Switching Neighbours Graph

- Node: CCZ class
- Edge: SN between classes
- Labels: Number in the DB

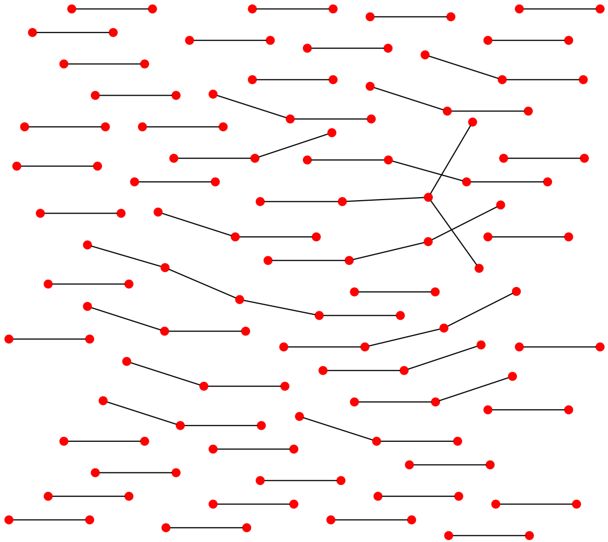
- 0: The cube
- 4: The Kim mapping
- 13: Only non CCZ-quadratic



For $\mathbb{F}_2^7$ - CCZ-Switching Neighbours Graph

- Node: CCZ-Quadratic Class
- Edge: SN between functions

About 10% have switching neighbours



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Size Concerns

- We have 3.8 millions new starting-points, but...
- The CCZ-classes of these functions contain roughly 300 EA-classes
- It takes roughly 0.6s to compute the switching neighbours of a 8-bit function

- $300 \times 0.6 \times 3.8 \times 10^6 \approx 7917$days of computations

For $\mathbb{F}_2^8$ - Experiments on Quadratic Functions

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However

These functions are all CCZ-quadratic and none is CCZ-equivalent to a bijection

For $\mathbb{F}_2^8$ - As is the tradition

[0,0,180,132,180,244,96,16,180,188,76,116,118,62,238,150,180,141,196,205,
103,30,119,62,144,161,172,173,53,68,105,40,107,243,216,112,156,68,79,167,
70,214,185,25,199,23,88,184,225,64,150,7,113,144,102,183,92,245,103,254,
186,83,225,56,180,185,152,165,58,119,118,11,148,145,244,193,108,41,108,
25,22,34,254,250,255,139,119,51,166,154,2,14,57,69,253,177,169,60,130,
39,100,177,47,202,16,141,119,218,171,118,172,65,53,153,218,70,159,115,
16,204,28,184,191,43,192,36,3,215,223,156,22,101,115,112,218,233,131,
200,6,125,89,82,188,135,75,49,70,12,128,186,237,231,135,245,198,132,
58,8,27,25,50,233,252,23,221,70,115,216,247,36,117,150,110,253,140,47,
152,122,146,64,16,178,122,232,205,39,139,81,51,153,21,143,66,12,19,109,
212,218,229,219,138,204,151,225,106,108,23,33,192,183,85,18,49,6,196,195,
152,231,65,14,31,32,166,169,217,15,143,105,12,154,58,156,136,86,146,124,
43,181,81,255,101,138,247,40,215,120,37,186,164,67,122,173,96,199,222,73]

sboxU v2 advert

- C++ optimized code
- Databases in tinySQL available for $n = 6$ and $n = 7$
- All the code needed to replicate our experiments
- Many more for S-box analysis

https://github.com/lpp-crypto/sboxU

- A lot of people involved from COSMIQ: Léo, Merlin, Guilhem, me
- And from faraway: Jules, Xavier, Baptiste G.
- Improved structure: it will be easier to contribute, help !
- What do YOU want to see implemented ?

Admissible Mapping and LAT

Linear Approximation Table (LAT)

The LAT is the table of the values $W_F(a, b) = \sum_{x \in \mathbb{F}_2^n} (-1)^{a \cdot x + b \cdot F(x)}$, where $\cdot$ is the usual scalar product over $\mathbb{F}_2^n$.

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- $\mathcal{V} := \{(x,0), x \in \mathbb{F}_2^n\}$
- The set of Walsh Zeroes $\mathcal{Z}_F := \{(\alpha,\beta) \mid W_F(\alpha,\beta) = 0\} \cup \{(0,0)\}$
- $\mathcal{V} \subset \mathcal{Z}_F$

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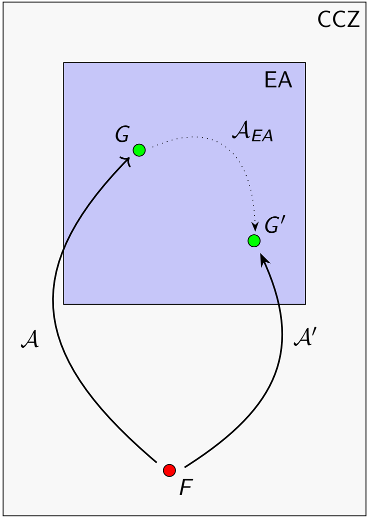
Admissibility Criterion [CP19]

The mapping $\mathcal{A}$ is admissible for F if and only if $\mathcal{A}^\top(\mathcal{V}) \subset \mathcal{Z}_F$.

Filtering with Walsh Zeroes



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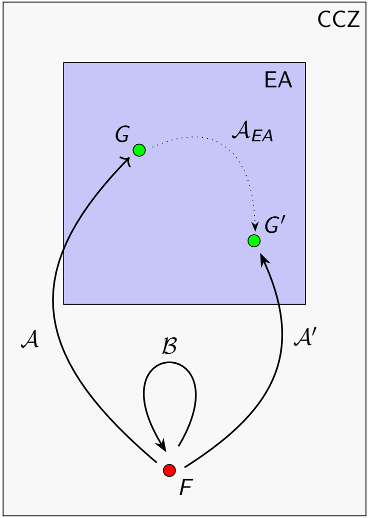

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G and G' are EA-equivalent if and only if there exists $\mathcal{B} \in \mathrm{Aut}(F)$ such that $\mathcal{B}^\top(V) = V'$.

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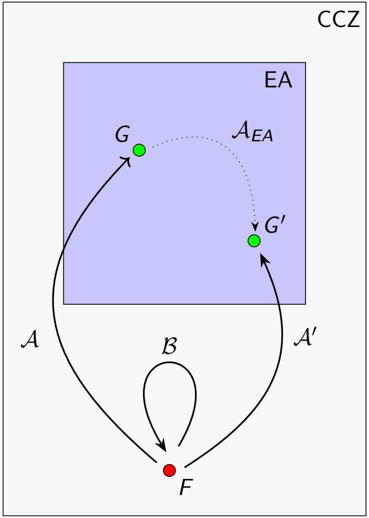

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Advantages

- Easier to test
- We already have an algorithm to search for the V [BPT19]

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Filtering Algorithm

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Since we have almost only quadratic functions

How to compute the automorphism group Aut of quadratic functions ?

$\mathrm{Aut}(F) = \mathrm{Aut}_{\mathrm{EA}}(F)$

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Theorem (CCZ equivalence of quadratic APN)

Let $n \geqslant 4$. Let $F, G : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be quadratic APN mappings. Then:

(i) Any automorphism of F is an EA-mapping: $\mathrm{Aut}(F) = \mathrm{Aut}_{\mathrm{EA}}(F)$ [KZ21]

(ii) if $\mathcal{A}$ is an admissible mapping satisfying $\mathcal{A}(\Gamma_F) = \Gamma_G$, then $\mathcal{A}$ is an EA-mapping.

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There is only one EA-class of quadratic functions in a CCZ-class, like it is for x^3

$\mathrm{Aut}(F)$ and $\mathrm{Aut}(\pi_F)$

Definition (Ortho-derivative)

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a quadratic APN function. The *ortho-derivative* of F is the function $\pi_F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ where $\pi_F(0) = 0$ and where for any $a \in (\mathbb{F}_2^n)^*$, $\pi_F(a)$ is the only non-zero element satisfying:

$$\pi_F(a) \cdot \big(F(x+a) + F(x) + F(a) + F(0)\big) \;=\; 0 \quad \forall\, x \in \mathbb{F}_2^n\,. \tag{1}$$

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$\mathrm{Aut}(F)$ and $\mathrm{Aut}_{\mathrm{LE}}(\pi_F)$ [CCP22]

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a quadratic APN function, if $F = B \circ F \circ A + C$, then π_F satisfies $\pi_F = B^T \circ \pi_F \circ A^{-1}$.

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The LAT of π_F looks like the LAT of random functions: they have large and 'rarer' coefficients.

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The LAT of the Kim mapping and its ortho-derivative

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- Coefficients of the same value are mapped to each other
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- Very fast algorithm for ortho-derivatives in our experiments

Computing the Group of Automorphisms

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Available in sboxU

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