Exploring the Set of APN Functions in Practice

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What Am I Talking about ?

Boolean Functions

Topic dear to Pascale, Anne, Léo, Shibam, Merlin and me

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Related to cryptography, more on that in a few minutes

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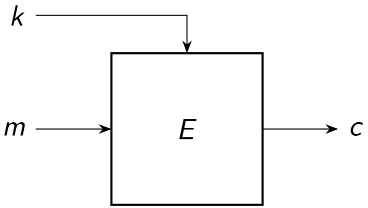
SboxU: A library for S-box analysis

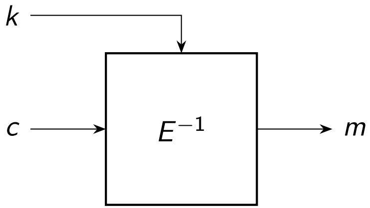
Important development project, with many team members involved, more on that throughout the presentation and a proper advert at the end

Definition of Block Ciphers

Block Cipher

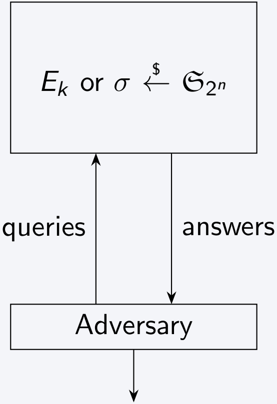
A family of keyed bijections $(E_k)_{k \in \mathcal{K}}$, $E_k : \mathbb{F}_2^n \to \mathbb{F}_2^n$





Ideal Security: Indistinguishability

The Indistinguishability Game



Was it drawn from $(E_k)_{k \in \mathcal{K}}$?

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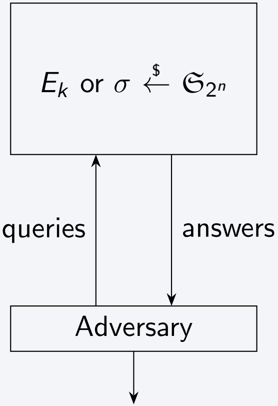

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- To build security proof on top of block-ciphers
- To avoid stronger attacks like key recovery

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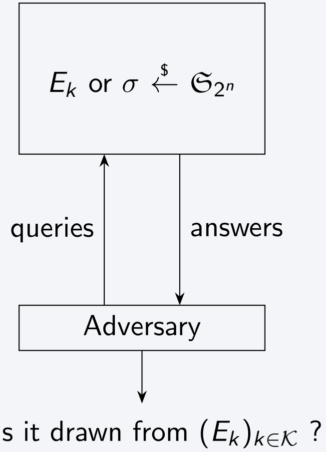

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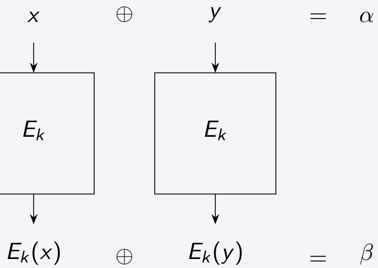
How to find a Distinguisher

What property would help to win the indistinguishability game ?

Differential Cryptanalysis

Principle (Biham and Shamir 91)



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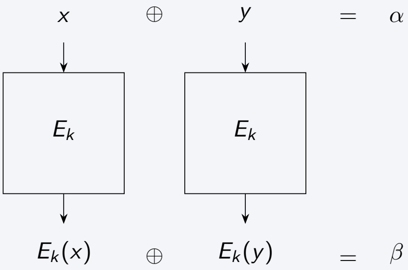

$\hookrightarrow$ How does the number of solutions $\#\{x \in \mathbb{F}_2^n \mid E_k(x + \alpha) + E_k(x) = \beta\}$ varies ?

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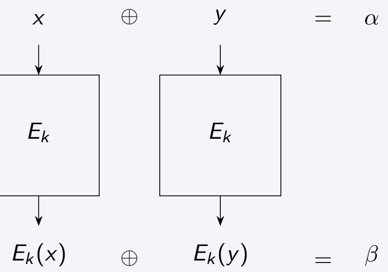

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For $\alpha \in \mathbb{F}_2^n \setminus \{0\}$ and $\beta \in \mathbb{F}_2^n$,
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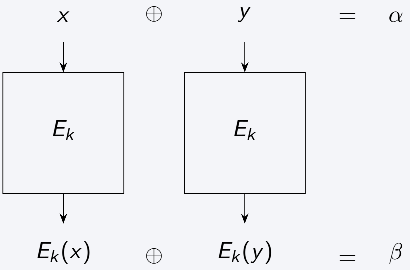

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Differential Distinguisher

A pair $(\alpha, \beta) \in \mathbb{F}_2^n \setminus \{0\} \times \mathbb{F}_2^n$, such that $E_k(x+\alpha) + E_k(x) = \beta$ has many solution in x

How to resist Differential Cryptanalysis ?

We start from the component level:

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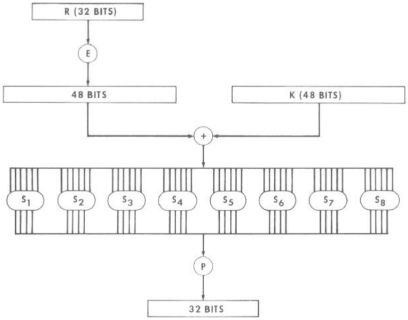

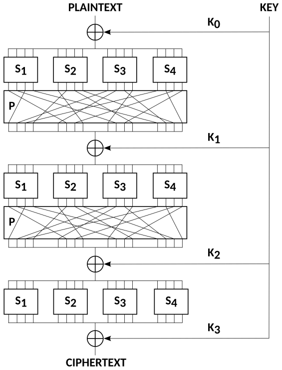

APN Functions

Definition (APN Function)

A function $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is *Almost Perfect Non-linear* (APN) if $F(x + a) + F(x) = b$ has at most 2 solutions for all $a \neq 0, b$.

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- **Almost**: in characteristic 2, we have an even number of solutions
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We need more properties for cryptography

- Sometimes, a S-box needs to be bijective
- It is usually better for it to have a higher degree

The Big APN Problem

Dillon Permutation $(n = 6)$

```
[ 0 54 48 13 15 18 53 35]
[25 63 45 52  3 20 41 33]
[59 36  2 34 10  8 57 37]
[60 19 42 14 50 26 58 24]
[39 27 21 17 16 29  1 62]
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- It was found *exploring* the set of APN functions starting from the Kim mapping:

$$x^3 + x^{10} + u\,x^{24}, \text{with } \langle u \rangle = \mathbb{F}_2^n \setminus \{0\}$$

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The Big APN Problem

Are there APN bijections when n is even and $n > 6$?

- x^3 is a bijection when n is odd

Goal

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Can we replicate the results of Dillon in dimension 8 by *exploring* the set of APN functions ?

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$\hookrightarrow$ What do we mean by exploration :

1. The computation of the **CCZ class**
2. The computation of **Switching Neighbours**

Definitions

$F : \mathbb{F}_2^n \to \mathbb{F}_2^n$, $f : \mathbb{F}_2^n \to \mathbb{F}_2$

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A quadratic Cubic example

$F(x) = x^3 = x^2 \cdot x$ is APN and has univariate degree 3 and algebraic degree 2

Equivalence Relations - 1

Extended-Affine Equivalence

F and G are *EA-equivalent* if there exists A, B affine permutations over $\mathbb{F}_2^n$ and C an affine mapping over $\mathbb{F}_2^n$ such that $G = B \circ F \circ A + C$. In terms of graph:

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Such affine permutation over $\mathbb{F}_2^{2n}$ is called an **EA-mapping**.

For all EA-mappings

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The cube again

$x^3 + L(x)$, $x^6 = (x^2)^3$ are EA-equivalent to x^3

Equivalence Relations - 2

Carlet-Charpin-Zinoviev (CCZ) Equivalence

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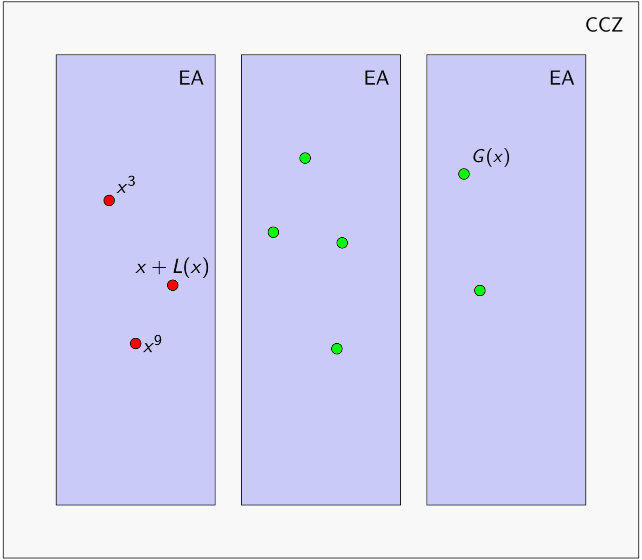
- $G(x) = x^3 + (x^2 + x + 1)tr(x^3)$ is CCZ-equivalent to x^3 but not EA-equivalent

Equivalence - Summary



From now on, we assume that affine mappings are linear for the sake of simplicity

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A first lower bound

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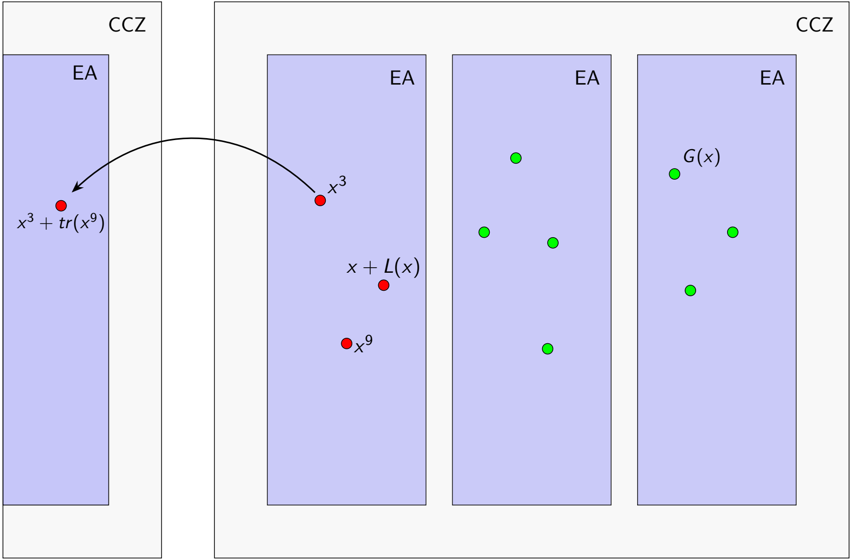
- And they conjecture there are more !

What is a new function ?



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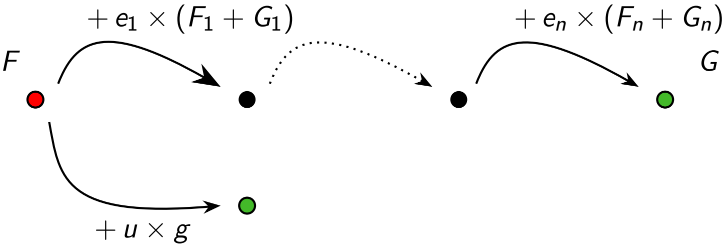

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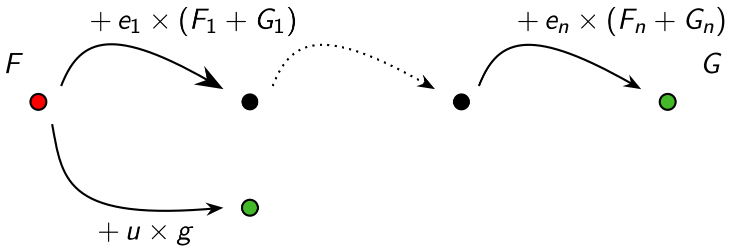
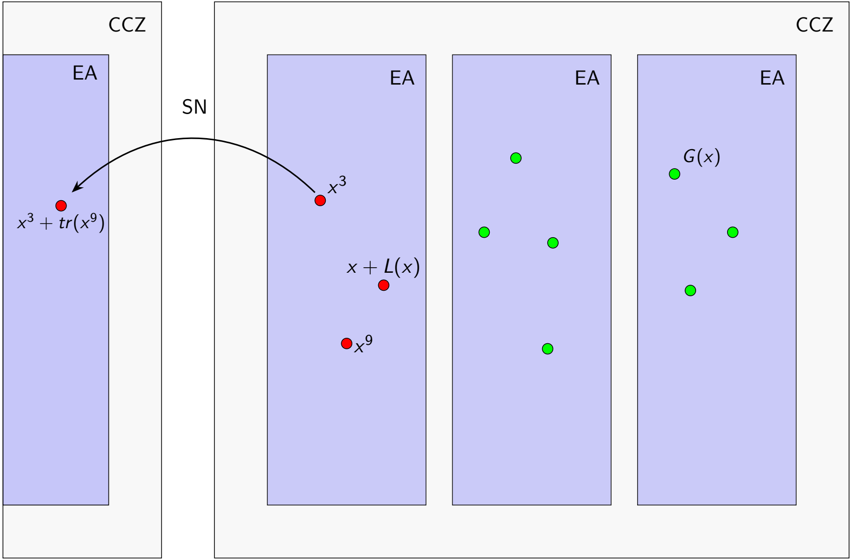


- Used to find the only known non-CCZ-quadratic APN in dimension 6 [EP09].
- Example: $x^3 + 1 \times tr(x^9)$, with $tr(x) = \sum_{i=0}^{n-1} x^{2^i}$

Exploration - Summary



Can we do it starting from the 3.8 millions function in dimension 8 ?

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- We provide algorithms to efficiently explore the CCZ-class
- We provide theoretic improvements and algorithms to compute Switching Neighbours
- We provide public C++ implementations with SAGE bindings for all these algorithms

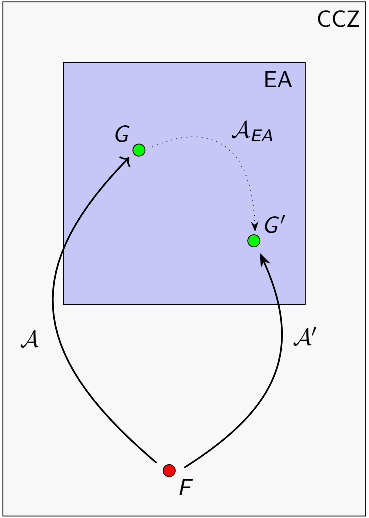
$\hookrightarrow$ From now on, everything is new !

Rest of the Talk

1. **Exploring the CCZ class**
2. **Exploring with Switching Neighbours**
3. **Experimental Results**

Exploring the CCZ class

Why is it harder than it seems ?



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- Naively: $\mathcal{A}_{EA} = \mathcal{A} \circ \mathcal{A}'^{-1}$

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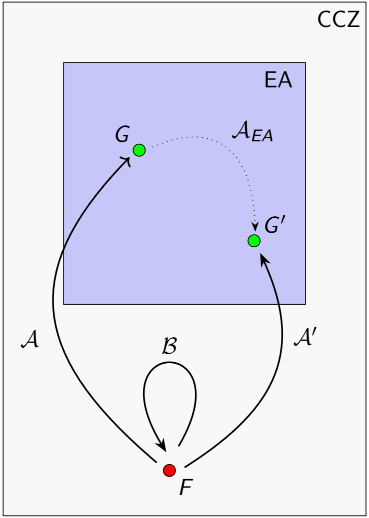

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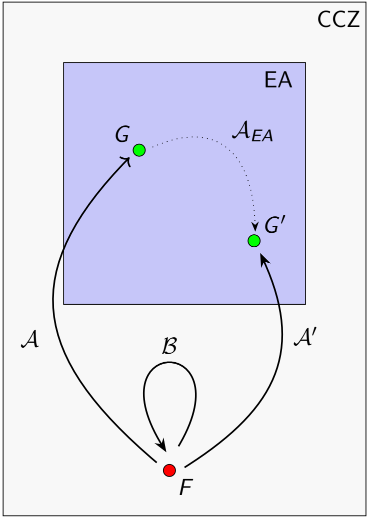

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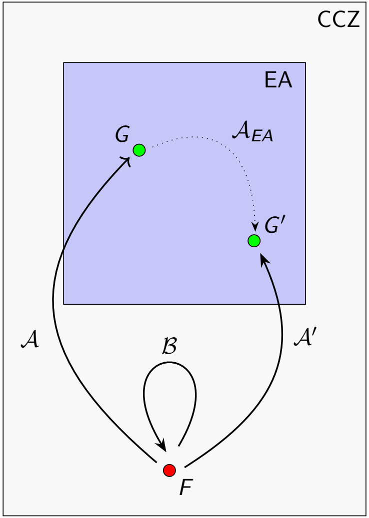

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Conclusion

We have to compute $\mathrm{Aut}(F)$ to quotient, but can we do it efficiently ?

Admissible Mapping and LAT

Linear Approximation Table (LAT)

The LAT is the table of the values $W_F(a,b) = \sum_{x \in \mathbb{F}_2^n} (-1)^{a \cdot x + b \cdot F(x)}$, where $\cdot$ is the usual scalar product over $\mathbb{F}_2^n$.

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- $\mathcal{V} := \{(x, 0), x \in \mathbb{F}_2^n\}$
- The set of Walsh Zeroes $\mathcal{Z}_F := \{(\alpha, \beta) \mid W_F(\alpha, \beta) = 0\} \cup \{(0, 0)\}$
- $\mathcal{V} \subset \mathcal{Z}_F$

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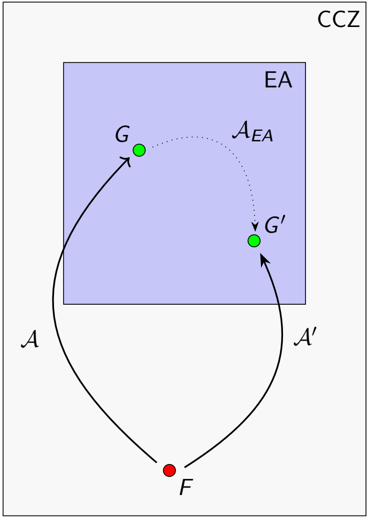
Admissibility Criterion [CP19]

The mapping $\mathcal{A}$ is admissible for F if and only if $\mathcal{A}^\top(\mathcal{V}) \subset \mathcal{Z}_F$.

Filtering with Walsh Zeroes



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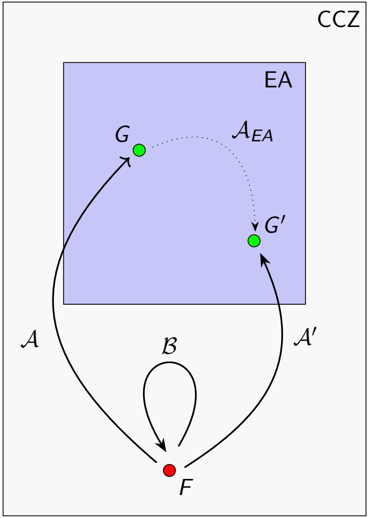

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Proposition (refined from [CP19])

G and G' are EA-equivalent if and only if there exists $\mathcal{B} \in \mathrm{Aut}(F)$ such that $\mathcal{B}^\top(V) = V'$.

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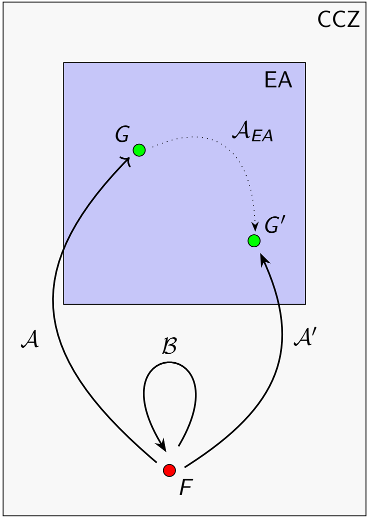

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Advantages

- Easier to test
- We already have an algorithm to search for the V [BPT19]

Filtering EA-equivalent functions

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Since we have almost only quadratic functions

How to compute the automorphism group Aut of quadratic functions ?

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Theorem (CCZ equivalence of quadratic APN)

Let $n \geqslant 4$. Let $F, G : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be quadratic APN mappings. Then:

(i) Any automorphism of F is an EA-mapping: $\mathrm{Aut}(F) = \mathrm{Aut}_{\mathrm{EA}}(F)$ [KZ21]

(ii) if $\mathcal{A}$ is an admissible mapping satisfying $\mathcal{A}(\Gamma_F) = \Gamma_G$, then $\mathcal{A}$ is an EA-mapping.

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There is only one EA-class of quadratic functions in a CCZ-class, like it is for x^3

$\mathrm{Aut}(F)$ and $\mathrm{Aut}(\pi_F)$

Definition (Ortho-derivative)

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a quadratic APN function. The *ortho-derivative* of F is the function $\pi_F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ where $\pi_F(0) = 0$ and where for any $a \in (\mathbb{F}_2^n)^*$, $\pi_F(a)$ is the only non-zero element satisfying:

$$\pi_F(a) \cdot \big(F(x+a) + F(x) + F(a) + F(0)\big) \; = \; 0 \quad \forall \, x \in \mathbb{F}_2^n \,. \tag{1}$$

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$\mathrm{Aut}(F)$ and $\mathrm{Aut}_{\mathrm{LE}}(\pi_F)$ [CCP22]

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a quadratic APN function, if $F = B \circ F \circ A + C$, then π_F satisfies $\pi_F = B^T \circ \pi_F \circ A^{-1}$.

LAT of π_f and Computing $\mathrm{Aut}(\pi_F)$

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- Very fast algorithm for ortho-derivatives in our experiments

Computing the Group of Automorphisms

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Available in sboxU

`graph_automorphisms_of_apn_quadratic`

Exploring with Switching Neighbours

1. Exploring the CCZ class

2. Exploring with Switching Neighbours

3. Experimental Results

The Switching Neighbors Criterion

Switching Neighbour Criterion [EP09]

If F is an APN function, $F + u \times g$ is an APN function if and only if :

$$g(x) + g(x + a) + g(y) + g(y + a) = 0$$

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Why is it a linear problem ?

The criterion boils down to $v_{x,y,a} \cdot lut(g) = 0$ with

- $lut(g) = (g(0), ..., g(2^n - 1)) \in (\mathbb{F}_2)^{2^n}$
- $v_{x,y,a} = (0, ..., 0, 1, 0, ..., 0, 1, 0, ..., 0, 1, 0, ..., 0, 1, 0, ..., 0) \in (\mathbb{F}_2)^{2^n}$

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If M_u is the matrix composed of $v_{x,y,a}$, we need to compute $Ker(M_u)$

Some Size Problems

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$\hookrightarrow$ It is not going to be feasible with just better implementations

Switching Neighbours and EA equivalence

The set switches $\mathrm{SW}_F(u)$

The set of boolean functions g for which $F + u \times g$ is APN is noted $\mathrm{SW}_F(u)$. We have that:

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EA-invariance

- We only need to compute the switching neighbours of one function in the EA class !
- Can be turned into a powerful invariant

Filtering Switching Neighbours

Affine Functions

- $\mathrm{Aff}(n) \subset \mathsf{SW}_F(u)$ is of dimension $n + 1$
- $\forall g \in \mathrm{Aff}(n),\ F + u \times g$ is EA-equivalent to F

Filtering Switching Neighbours

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Coordinates

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Non-Trivial Switching Neighbours

We can decompose $\mathrm{SW}_F(u)$ in the following direct sum:

$$\mathrm{SW}_F(u) = \mathrm{SW}_F^{nt}(u) \oplus \mathrm{Aff}(n) \oplus \langle F_1, \ldots, F_n \rangle \cap \mathrm{SW}_F(u)$$

For such vector space $\mathrm{SW}_F^{nt}(u)$, we call $F + u \times \mathrm{SW}_F^{nt}(u)$ a set of non-trivial switching neighbours of F.

Computing Switching Neighbours in practice

Rephrasing of Edel and Pott's Criterion

- Let M_u be composed of $v_{x,y,a} = (0, ..., 0, 1, 0, ..., 0, 1, 0, ..., 0, 1, 0, ..., 0, 1, 0, ..., 0)$ with: $F(x) + F(x + a) + F(y) + F(y + a) = u$
- $\mathrm{SW}_F(u) = Ker(M_u)$
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- Draw lines in $v_{x,y,a}$ at random to get a matrix M such that $Ker(M) \supset Ker(M_u)$

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As it turns out, M is of maximum rank much faster than expected. For $n = 8$ we get $Ker(M) = Ker(M_u)$ in roughly 2.5×2^n random equations.

Switching Neighbours - Summary

Our Improvements

- We have reduced by a factor 2^{3n} the number of switching neighbours per function that we need to check

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Available in `sboxU`

- `non_trivial_sn`
- We also have very efficient implementations of linear algera !

Experimental Results

1. Exploring the CCZ class

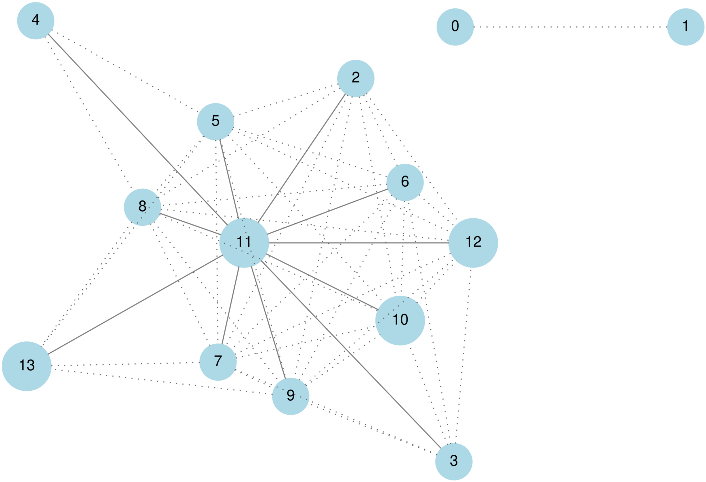
2. Exploring with Switching Neighbours

3. **Experimental Results**

For $\mathbb{F}_2^6$ - CCZ-SN Graph

- Node: CCZ class
- Edge: SN between classes
- Labels: Number in the DB

- 0: The cube
- 4: The Kim mapping
- 13: Only non CCZ-quadratic



For $\mathbb{F}_2^6$ - Graph Summary

CCZ-SN components

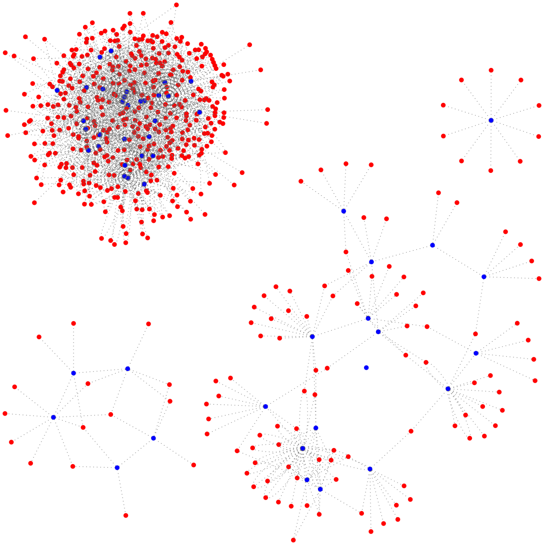
- The component of the cube - labelled 0
- The component of the Kim mapping - labelled 4
- The first graph is the identical when we consider only quadratic functions

Class 11

$F(x) = x^9 + u^4\,(x^{10} + x^{18}) + u^9\,(x^{12} + x^{20} + x^{40})$ is connected to all other CCZ-classes in its component and it is the only one !

For $\mathbb{F}_2^6$ - CCZ-Class 11

- Blue node: EA class in 11
- Red node: EA class **not** in 11
- Edge: SN EA-classes



For $\mathbb{F}_2^6$

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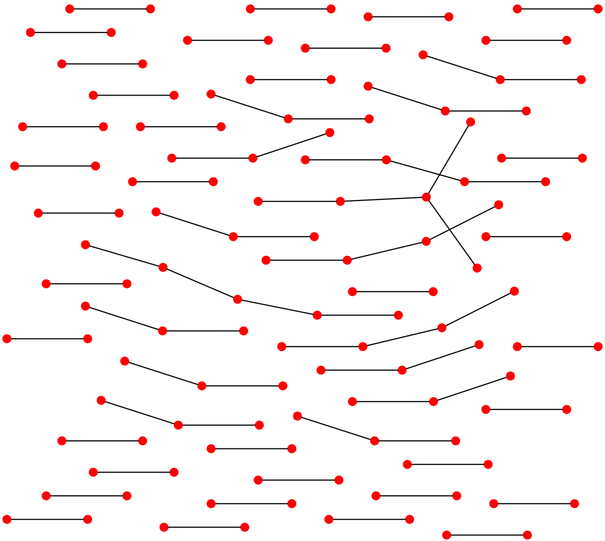
Conclusion

- All EA-classes are stored in a tinySQL database with invariants to help to test new functions
- If new APN functions exist, they are in another component
- Lightning fast performances to do experiments !

For $\mathbb{F}_2^7$ - Quadratic only

- Node: Quadratic function
- Edge: SN between functions

The size of the components does not seem to scale



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Experimental Results

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- New : The union of their CCZ-classes contains a total of 82,409 EA-classes
- Only 9890 functions have switching neighbours - 12 % of the entire set
- No new functions were found

Conclusion

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- Possibly many other small CCZ-SN components
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However

These functions are all CCZ-quadratic and none is CCZ-equivalent to a bijection

For $\mathbb{F}_2^8$ - As is the tradition

[0,0,180,132,180,244,96,16,180,188,76,116,118,62,238,150,180,141,196,205,
103,30,119,62,144,161,172,173,53,68,105,40,107,243,216,112,156,68,79,167,
70,214,185,25,199,23,88,184,225,64,150,7,113,144,102,183,92,245,103,254,
186,83,225,56,180,185,152,165,58,119,118,11,148,145,244,193,108,41,108,
25,22,34,254,250,255,139,119,51,166,154,2,14,57,69,253,177,169,60,130,
39,100,177,47,202,16,141,119,218,171,118,172,65,53,153,218,70,159,115,
16,204,28,184,191,43,192,36,3,215,223,156,22,101,115,112,218,233,131,
200,6,125,89,82,188,135,75,49,70,12,128,186,237,231,135,245,198,132,
58,8,27,25,50,233,252,23,221,70,115,216,247,36,117,150,110,253,140,47,
152,122,146,64,16,178,122,232,205,39,139,81,51,153,21,143,66,12,19,109,
212,218,229,219,138,204,151,225,106,108,23,33,192,183,85,18,49,6,196,195,
152,231,65,14,31,32,166,169,217,15,143,105,12,154,58,156,136,86,146,124,
43,181,81,255,101,138,247,40,215,120,37,186,164,67,122,173,96,199,222,73]

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Example: CCZ-class of x^3

- For $n = 6$, there are **exactly** 3 EA-classes
- For $n = 7$, there are **exactly** 3 EA-classes
- For $n = 8$, there are **exactly** 2 EA-classes

`sboxU` v2 advert

- C++ optimized code
- Databases in tinySQL available for $n = 6$ and $n = 7$
- All the code needed to replicate our experiments
- Many more for S-box analysis

https://github.com/lpp-crypto/sboxU

- A lot of people involved from COSMIQ: Léo, Merlin, Guilhem, me
- And from faraway: Jules, Xavier, Baptiste G.
- Improved structure: it will be easier to contribute, help !
- What do YOU want to see implemented ?

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